

A few of Michel Hénon's contributions to dynamical astronomy

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Abstract

This article reviews Michel Hénon's contributions to a diverse set of problems in astrophysical dynamics, including violent relaxation, Saturn's rings, roundoff error in orbit integrations, and planet formation.

I only met Michel Hénon once, but I followed his tracks many times. I first encountered his work in the early 1970s, when I was a graduate student beginning to learn stellar dynamics. At that time the subject was just beginning to blossom, and the selection of textbooks and reviews was quite limited ([5], [36], [33]). Thus I was delighted to discover MH's Saas-Fee lectures on "Collisional dynamics of spherical stellar systems" [21], and I soon learned that *any* paper by MH was worth reading, even if I had to use my very limited French. It is sometimes said that papers by great physicists such as Maxwell and Einstein are easier to understand than papers by physicists who are good but not great, and MH's papers had a similar quality, in that simple logic led to profound conclusions. Because of space and time constraints, I cannot describe all of his work that interested and impressed me, so I have tried to present a selection that illustrates the unusual diversity of his contributions to astrophysical dynamics, and focuses on topics in which we shared a common interest.

Violent relaxation. I

The evolution of a cluster of stars or other self-gravitating N-body system can crudely be divided into two phases. (i) On timescales short compared to the relaxation time t_{relax} the dynamics is that of a collisionless system, in which each star moves under the influence of the smooth gravitational field generated by its $N - 1$ siblings. The evolution is described by the collisionless Boltzmann

equation¹ and the Poisson equation, relating the phase-space density $f(\mathbf{x}, \mathbf{v}, t)$ and the gravitational potential $\Phi(\mathbf{x}, t)$,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \quad \nabla^2 \Phi = 4\pi G \int d\mathbf{v} f. \quad (1)$$

A collisionless N-body system that is out of equilibrium will evolve rapidly to an approximately stationary solution of the collisionless Boltzmann equation. In cosmology this process is sometimes called “virialization” since the stationary system satisfies the time-independent virial theorem $2K + W = 0$, where K and W are the kinetic and potential energy. Virialization is a rapid process: it takes only a few times the crossing time $t_{\text{cross}} = R/V$, where R and V are a typical radius and velocity in the system. The evolution is driven by large-scale oscillations in the gravitational potential, which rapidly damp through phase mixing. (ii) On timescales long compared to t_{relax} the evolution is driven by two-body encounters between each star and its neighbors; the system evolves slowly along a sequence of approximate solutions of the collisionless Boltzmann equation; during this evolution the central density grows while stars escape from the outer parts². The ratio of the relaxation and crossing times is [3](e.g., eq. 1.38).

$$\frac{t_{\text{relax}}}{t_{\text{cross}}} \simeq \frac{0.1N}{\ln N}. \quad (2)$$

In the early 1960s N-body integrations of self-gravitating systems were still extremely crude: a complete literature survey up to 1964, so far as I know, consists of [31], [29], [30], and [1], all with $N \leq 100$. The results from simulations of such small systems are difficult to interpret, not just because of statistical fluctuations but also because equation (2) implies that the relaxation and crossing times are not well-separated unless $N \gg 100$, so the two phases of evolution are not distinct.

MH’s elegant idea [18], was to restrict the N-body system to spherical symmetry, replacing the point masses by spherical shells (the idea is originally presented by [4], who in turn attributes it to George Gamow). Shell j has mass m_j , radius $r_j(t)$, and conserved angular momentum per unit mass L_j . The equations of motion are

$$\frac{d^2 r_j}{dt^2} = \frac{L_j^2}{r_j^3} - \frac{GM_j}{r_j^2}, \quad M_j = \sum_{r_k < r_j} m_k + \frac{1}{2}m_j. \quad (3)$$

As a result of this simple ansatz, (i) two-body relaxation is greatly reduced, because the force field from a spherical shell is much smoother than the force field

¹Sometimes also called the Liouville equation, the Vlasov equation, or other names; but MH had strong opinions on the appropriate name to use [24], and I agree with him.

²The first substantive discussion of this process was in MH’s doctoral thesis at the University of Paris [17].

from a point; (ii) the equations of motion are simpler; for example one follows one coordinate instead of three and the equations can be solved analytically between shell crossings; (iii) the simulation preserves the spherical symmetry that one expects to see in large- N systems.

MH followed several systems with $N \leq 100$, starting from “top hat” initial conditions (uniform density inside a spherical volume), with a Maxwellian velocity distribution and initial virial ratio $-2K/W = 0.5$. A later calculation with $N = 1000$ is described in [19]. These crude simulations yielded the first clear description of the process of virialization. MH showed that the final stationary states depended somewhat on the initial conditions but all shared certain properties, including the presence of a dense central core, an extended envelope in which the density decayed as $\rho(r) \sim r^{-4}$, and a velocity-dispersion tensor that was mainly radial in the envelope. This work was the precursor to Lynden-Bell’s (1967) seminal work on virialization (which he termed “violent relaxation”), and to the “basis-function” or “self-consistent field” N -body codes [42, 28, 39] that are now widely used to study galaxy mergers, galaxy stability [20], and other collisionless collective processes.

Violent relaxation. II

In the mid-1980s I became interested in whether arguments from statistical mechanics could be used to predict the stationary solutions of the collisionless Boltzmann equation that resulted from violent relaxation. Typically, attempts to do this, e.g. [67], lead to solutions with infinite mass, so my goal was to address a more modest question: if $f_i(\mathbf{x}, \mathbf{v})$ and $f_f(\mathbf{x}, \mathbf{v})$ are the phase-space distribution functions of the stellar system before and after violent relaxation, what constraints can be placed on f_f given f_i , other than mass, energy, and angular-momentum conservation?

Violent relaxation can be thought of as a Markov process in which mass elements are shuffled among cells in phase space. By definition, any Liapunov functional $L[f]$ of the Markov process decreases at each step of the process; thus, if $L[f_f] \leq L[f_i]$ for all Liapunov functionals, then f_f is said to be more mixed than f_i (e.g. [43, 13]). I was able to find a simple criterion for this partial ordering—more accurately, I was able to look it up in [14]—which was later simplified further by [8]: define a one-parameter family of functionals

$$D_\phi[f] = \int d\mathbf{x} d\mathbf{v} \max[f(\mathbf{x}, \mathbf{v}) - \phi, 0]. \quad (4)$$

Then f_f is more mixed than f_i if and only if $D_\phi[f_f] \leq D_\phi[f_i]$ for all ϕ . One corollary of this approach is that the standard Boltzmann entropy $S = -\int d\mathbf{x} d\mathbf{v} f \log f$ plays no special role in violent relaxation, since $-S$ is only one

of many possible Liapunov functionals of the form $\int d\mathbf{x}d\mathbf{v} C(f)$ where C is a convex function.

While preparing that work for publication, I learned to my surprise that MH and Donald Lynden-Bell had worked on the same problem two decades earlier, without publishing their results. I got copies of their notes and correspondence and discovered that they had obtained almost all the same results that I had—although their achievement was much more impressive since most of the work in the statistical mechanics literature that I used had not been published at that time. Moreover, MH had the following beautiful physical analogy to explain the result (4): Consider a hill in which the height above sea-level is $h_i(x, y)$. An engineer wishes to transform the hill to a differently shaped one, with a new topography $h_f(x, y)$. We assume that there is only one summit or local maximum in both $h_i(x, y)$ and $h_f(x, y)$. The engineer has a bulldozer that can scrape off earth from the hill and push it downhill; however, the bulldozer is not powerful enough to push the material uphill. It is not hard to see that the engineer can succeed at this task if and only if there is more earth above every height H in the initial hill than in the final hill, that is, if $D_H[h_f] \leq D_H[h_i]$, which is precisely analogous to condition (4).

After some correspondence the three of us agreed to publish the paper together [41]; not only was I proud to be able to collaborate with two of my scientific heroes, but in the course of our discussions MH suggested that we try a new communication medium called e-mail and sent me my first e-mail message!

I think the main impact of this work has been to dampen the enthusiasm of theorists for explaining “universal” profiles resulting from virialization or violent relaxation, such as the famous NFW profile [34], in terms of maximum-entropy arguments: such arguments cannot give a unique final state unless one first shows why some particular entropy $-\int d\mathbf{x}d\mathbf{v} C(f)$ is the only relevant one.

Roundoff error

Roundoff error has always plagued numerical explorations of dynamical systems, in particular long integrations of planetary systems (e.g. [16]). The problem with roundoff error is not so much that phase-space positions and other parameters cannot be represented exactly, but rather that it accumulates with time; in a prescient paper [35] pointed out that roundoff errors in planetary positions should grow with time at $t^{3/2}$. In fact, Newcomb was over-optimistic: unless one is very careful the errors in modern computations tend to grow at t^2 . The growth of roundoff error is inevitable in floating-point arithmetic because floating-point operations are many-to-one maps rather than one-to-one maps. As computers become faster and integration algorithms became more accurate,

roundoff error has played a growing role in the error budget of studies of the long-term behavior of dynamical systems.

Perhaps the most important advance in controlling roundoff error was the introduction in 1985 of the IEEE Standard for floating-point arithmetic (IEEE 754), which has now been adopted (more or less) in most compilers. Among its features, IEEE 754 requires that all arithmetic operations are rounded to the nearest number that can be represented exactly in the computer, and in the case of ties rounded to the nearest number whose least significant bit is even. IEEE 754 helps us to manage the disease of roundoff error but does not cure it.

MH studied this problem with his student Françoise Rannou [38]. First, they reduced the problem to its simplest form by studying an area-preserving map in a compact phase space of two dimensions rather than a Hamiltonian system, a technique made famous a few years early by MH's work with another student, Carl Heiles [26]. The map was chosen to exhibit both invariant curves and chaotic regions. They then modified the map slightly so it was a one-to-one map of integers onto integers—in other words, they changed the map into a permutation of points on a lattice—and studied the properties of the modified map using fixed-point arithmetic, which has no roundoff error. An additional advantage of the lattice map is that all orbits are periodic so the properties of the map can be characterized completely with finite computing resources.

The most striking of Rannou's conclusions was that the lattice maps bear a strong visual resemblance to the original floating-point maps, with invariant curves in the original map corresponding to periodic orbits with short cycles and chaotic orbits to long cycles. This and other findings, in her understated conclusions, “encourage the view that conventional computer studies are not seriously affected by roundoff errors”.

In the early 1990s I worked on lattice maps for Hamiltonian systems with David Earn, who was then a student at Toronto. Partway through the work, we were disappointed to learn—although by this time I should not have been surprised—that MH and Rannou had covered much of the ground long before us. We extended Rannou's work with more thorough numerical calculations, and were able to prove that symplectic maps from R^{2N} to itself could be replaced by maps that were the restriction to a lattice of a symplectic map that was “close” to the original map in a precise sense [9]; thus lattice maps can be thought of as replacing forward error analysis with backward error analysis. The lattice maps differ from the original symplectic map by a small but rapidly varying perturbation; as the lattice spacing shrinks the difference becomes smaller but it varies more and more rapidly. Thus the Kolmogorov-Arnold-Moser or KAM theorem does not guarantee that invariant tori in the original map will survive in the lattice map, no matter how small the spacing, although in practice the tori seemed to be quite robust. The question of whether integer or floating-point calculations are more faithful representations of the long-term behavior of

Hamiltonian systems remains open.

Perhaps the simplest demonstration of the long-term effects of roundoff error comes from the rotation map

$$x_{n+1} = x_n \cos \theta - y_n \sin \theta, \quad y_{n+1} = x_n \sin \theta + y_n \cos \theta, \quad (5)$$

with fixed angle θ and initial conditions $x_0 = 1$, $y_0 = 0$. The map should conserve $E_n \equiv x_n^2 + y_n^2$ but numerical computations generally show a linear drift, $E_n - 1 \propto n$, with the slope of the drift depending erratically on the rotation angle θ . Thomas Quinn and I investigated this map as a simple model for the growth of roundoff error in long orbit integrations [37], and we found recipes to reduce the drift to $E_n - 1 \propto n^{1/2}$. A central ingredient in these recipes is to correct for the fact that $\cos^2 \theta + \sin^2 \theta - 1$ is typically not exactly zero when evaluated with floating-point arithmetic.

[27] looked at this problem more deeply: in floating-point arithmetic, finding a value of θ for which $\cos^2 \theta + \sin^2 \theta - 1$ is close to zero is equivalent to the Diophantine problem of finding integers m and n such that $m^2 + n^2 = 2^{2p} + k$ where $p = 53$ for IEEE 754 double-precision arithmetic and $|k|$ is as small as possible. They were able to show that the best solutions have $k = 1$, although there are only eight for $0 \leq \theta \leq \pi/4$, and to give explicit algorithms for finding the many solutions for which $|k|$ exceeds unity but is still small.

Regrettably, MH's work on roundoff error has not received much attention. Most computational astrophysicists still adopt the convenient belief that "double precision is accurate enough". Often it is, but not always. Like many other arcane problems in theoretical computer science, the problems studied by MH in this subject may prove to be important only many years after they were first posed and solved.

Apples in a spacecraft

This controversy began with a thought experiment by the Nobel Prize winner Hannes [2]. Consider a spacecraft in a circular orbit around the Earth. The spacecraft is assumed to be synchronously rotating (the same side always faces the Earth). Now release a swarm of inelastically colliding objects ("apples") in the spacecraft; what will be their final state? Alfvén concluded that the apples would collect in a pile at the center of mass of the spacecraft.

The following argument is equivalent to Alfvén's, but simpler and more concise. Let m_i , \mathbf{r}_i , and \mathbf{v}_i , $i = 0, \dots, N$ be the masses, positions, and velocities of the spacecraft ($i = 0$) and the N apples, in an inertial frame. The energy and angular momentum of this collection are

$$E = \sum_{i=0}^N m_i \left(\frac{1}{2} v_i^2 - \frac{GM_\oplus}{r_i} \right), \quad \mathbf{L} = \sum_{i=0}^N m_i \mathbf{r}_i \times \mathbf{v}_i. \quad (6)$$

We extremize the energy at fixed angular momentum. Using a vector Lagrange multiplier $\boldsymbol{\lambda}$, the condition for an extremum is

$$0 = \delta E - \boldsymbol{\lambda} \cdot \delta \mathbf{L} = \sum_{i=0}^N m_i \left[\delta \mathbf{v}_i \cdot (\mathbf{v}_i - \boldsymbol{\lambda} \times \mathbf{r}_i) + \delta \mathbf{r}_i \cdot \left(\frac{GM_{\oplus}}{r_i^3} \mathbf{r}_i + \boldsymbol{\lambda} \times \mathbf{v}_i \right) \right]. \quad (7)$$

This requires

$$\mathbf{v}_i = \boldsymbol{\lambda} \times \mathbf{r}_i, \quad \lambda^2 = \frac{GM_{\oplus}}{r_i^3}, \quad \boldsymbol{\lambda} \cdot \mathbf{r}_i = 0; \quad (8)$$

in words, all of the orbits must be coplanar and circular, with the same orbital radius.

The importance of Alfvén’s thought experiment is that it suggests a way to collect small bodies such as planetesimals into dense agglomerations that might form planets (these were called “jet streams” by their proponents).

Some years later, [22] wrote a short but crushing reply, which began “Unfortunately, Alfvén’s reasoning is incorrect and the final state of the system is in reality rather different from what he predicts. This conclusion has apparently escaped notice so far, and Alfvén’s result continues to be cited uncritically. Thus it appears desirable to correct the record.” Although MH’s reasoning was physical rather than mathematical—there are no equations in his paper, compared to two dozen in Alfvén’s—it is simplest to describe in the context of the derivation above. The method of Lagrange multipliers finds an extremum in the energy at fixed angular momentum, but the extremum represented by equation (8) is a saddle point, not a minimum. The minimum energy state occurs when half the apples are on the floor of the spacecraft (the point closest to Earth) and the other half on the ceiling. Dissipation acts to disperse the radii of the apples, not to bring them together; Alfvén’s model is an argument against jet streams rather than in favor of them, and indeed this concept no longer plays a role in models of planet formation³.

Saturn’s rings

When NASA’s Voyager 1 and 2 spacecraft flew past Saturn in 1980 and 1981, they revealed that the planet’s famous rings were far more complex than previously suspected (see [12] for a post-Voyager review and [6] for a recent one). Among the puzzles emerging from the spacecraft data were the following: (i) Collisions between ring particles redistribute angular momentum so the ring

³In the early 1970s I was a graduate student in physics at Princeton, which then had a series of written general exams that had to be completed before starting a thesis. One of the problems on the exam in spring 1971 was to prove Alfvén’s result. I’m proud to report that my fellow students, like MH, recognized that the result was wrong, although only after the exam, and took pleasure in pointing this out to the professors.

spreads, much like the gaseous accretion disks in other astrophysical systems, and sharp radial features are washed out. However, Voyager revealed a rich spectrum of narrow gaps, ringlets, and other features on all scales down to the spacecraft resolution limit of ~ 10 km. (ii) The thickness of the rings can be estimated by photometric observations during the Earth's passages through the ring plane, which occur every 13 years. These suggest that the ring thickness is $H \sim 1$ km. However, inelastic collisions between the ring particles rapidly damp any motions normal to the ring plane, so the ring should be much thinner than the ring-plane-crossing observations imply.

MH's work on Saturn's rings began with a simple question: *is* there a typical size of a ring particle [23, 25]? Natural processes such as grinding, fragmentation by high-velocity impacts, or coagulation generally produce power-law distributions of particle size over several orders of magnitude, and such size distributions are seen in asteroids, meteorites, and debris on the lunar surface. Moreover the exponents of such power laws have a relatively narrow range: if the number of particles in a small radius range is

$$dN = \left(\frac{r_0}{r}\right)^\beta d\log r \quad (9)$$

then $\beta \simeq 1.8\text{--}3.6$ (e.g. [15]). Thus the ring particles may be better characterized by a power-law distribution than by a single size.

MH explored the implications of this ansatz. In the following discussion, we take dN in equation (9) to refer to the total number of ring particles in the radius range $d\log r$, and assume that this power law applies for all radii larger than r_{\min} , which is several orders of magnitude smaller than r_0 . With this normalization r_0 is approximately the size of the largest particle in the rings.

If the ring is flat and the centers of the particles lie in the ring's midplane, the apparent thickness H of the edge-on ring is given approximately by the diameter of the largest particles for which the optical depth of the edge-on ring is unity. This yields

$$H \left(\frac{r_0}{H}\right)^\beta \simeq R \quad (10)$$

where $R \simeq 10^{10}$ cm is the radius of the rings.

The ring particles can only efficiently scatter radiation of wavelength λ if $r \gtrsim \lambda/(2\pi)$. Pre-Voyager ground-based measurements of the radar cross-section and radio brightness temperature of the rings could therefore be used to constrain the distribution of ring particles on size scales of a few cm. They implied that the normal geometrical optical depth of the rings is of order unity for $r = r_r \simeq 4$ cm, or

$$r_r^2 \left(\frac{r_0}{r_r}\right)^\beta \simeq 2R\Delta R \quad (11)$$

where $\Delta R \simeq 4 \times 10^9$ cm is the radial range of the densest part of the rings. Equations (10) and (11) are sufficient to determine the parameters of the size

distribution⁴:

$$\beta = 3.1, \quad r_0 = 40 \text{ km.} \quad (12)$$

The ring particles orbit inside their Roche limit, given by

$$R_{\text{Roche}} = 1.523(M/\rho)^{1/3} = 1.31 \times 10^{10} \text{ cm} \left(\frac{0.9 \text{ g cm}^{-3}}{\rho} \right)^{1/3}, \quad (13)$$

where $M = 5.684 \times 10^{29} \text{ g}$ is the mass of Saturn and the particle density ρ is given relative to that of ice. The Roche limit is the orbital radius at which a fluid satellite will be tidally disrupted icy objects as large as r_0 can be held together in the tidal field of Saturn by their internal strength.

Since the exponent β is larger than 3, the ring mass is dominated by small particles rather than large ones. We have

$$M_{\text{ring}} = \frac{4}{3}\pi\rho \int d\log r r^3 \left(\frac{r_0}{r} \right)^\beta = \frac{4\pi}{3(\beta-3)}\rho r_0^\beta r_{\text{min}}^{3-\beta} = 1.1 \times 10^{22} \text{ g} \frac{\rho}{0.9 \text{ g cm}^{-3}} \left(\frac{1 \text{ cm}}{r_{\text{min}}} \right)^{0.1}. \quad (14)$$

The result depends only weakly on the poorly known minimum size r_{min} . This estimate of the ring mass is consistent with an independent estimate from the dispersion relation for density waves, $M_{\text{ring}} = (3 \pm 2) \times 10^{22} \text{ g}$ [10], although this may be an underestimate because density waves are not present in the densest parts of the rings.

The wide range of particle sizes in the rings has other consequences. Massive particles traveling on circular orbits gravitationally repel particles on nearby orbits: at each conjunction the gravitational force from the massive particle excites radial oscillations (eccentricity) in the smaller particle, and because the Jacobi constant is conserved the enhanced eccentricity results in a transfer of angular momentum from the inner particle to the outer one. The orbit-averaged gravitational torque between two masses m_1 and $m_2 \ll m_1$ on circular orbits of radii R and $R + d$, $d \ll r$, is [11]

$$T = 0.399 \frac{Gm_2^2 m_1 R^3}{Md^4} \quad (15)$$

where M is the mass of the central body. This expression is only valid if $d \gtrsim r_H$ where $r_H = R[(m_1 + m_2)/3M]^{1/3} \simeq R(m_1/3M)^{1/3}$ is the mutual Hill radius of the two particles; for smaller separations a rough approximation to the torque is obtained by replacing d by r_H in equation (15).

To analyze the effects of these torques, MH made a crude division of the ring particles, at a radius r_g to be determined below, into “big” and “small” particles. Big particles are massive enough that they can overcome the spreading of the

⁴The numbers given here can differ from the numbers given by MH by up to a factor of two or so. These differences are not important.

rings due to collisions and open up a gap around themselves, while small particles cannot. We can now estimate the integrated angular-momentum current from the gravitational torques between the small particles. Using equation (9) for the number dN of ring particles in a small size range, we can write the number of particles in a small range of size and orbital radius as $dN dR / \Delta R$ where ΔR is the ring width. The mass of a particle is $\frac{4}{3}\pi\rho r^3$. Since the torque (15) falls off rapidly with separation d we can approximate it as zero for $d \gtrsim r_H$ and equal to $\sim Gm_2^2 m_1 R^3 / (Mr_H^4)$ for $d \lesssim r_H$. Then the angular-momentum current is equal to the torque exerted by small particles inside R on small particles outside R , or

$$C_L^{\text{ring}} \simeq \frac{GR^3}{M\Delta R^2} \int_0^{r_g} dN(r) \int_0^r dN(r') \frac{m(r)^2 m(r')}{r_H^2(r)} \simeq \frac{GR\rho^{4/3}}{M^{1/3}\Delta R^2} M_{\text{ring}} r_0^\beta r_g^{4-\beta}; \quad (16)$$

here M_{ring} is the ring mass (eq. 14) and we have assumed $\beta > 3$. Since the ring is close to the Roche limit (eq. 13) $\rho \simeq M/R^3$ so this result simplifies to

$$C_L^{\text{ring}} \simeq \frac{G\rho}{\Delta R^2} M_{\text{ring}} r_0^\beta r_g^{4-\beta}. \quad (17)$$

A similar calculation yields the torque produced by a “big” particle of mass m on a uniform ring separated from it by a gap $d \gtrsim r_H$:

$$C_L^{\text{gap}} \simeq \frac{Gm^2 R^3}{M\Delta R d^3} M_{\text{ring}}. \quad (18)$$

This is the angular-momentum current across the gap, which in a steady state must equal the current C_L^{ring} through the ring on either side of the gap, so we find

$$d^3(r) \simeq \frac{m^2 R^3 \Delta R r_g^{\beta-4}}{\rho M r_0^\beta} \simeq \frac{r^6 r_g^{\beta-4} \Delta R}{r_0^\beta}; \quad (19)$$

in the last expression we have again set $\rho \simeq M/R^3$. The division between big and small particles is at the radius r_g where $d \sim r_H$, which yields $r_g \simeq (r_0^\beta / \Delta R)^{1/(\beta-1)} \simeq 1.5 \text{ km}$. The minimum gap size is $\sim 2r_g$ or a few kilometers, and the gap size scales with the size of the big particle as $d \propto r^2$. The number of big particles, and thus the number of gaps, is given by equation (9) as $N_{\text{gap}} = \simeq \int_{r_g}^{r_0} dN \simeq 8 \times 10^3$. The total width of the gaps is

$$\Delta R_{\text{gap}} \simeq \int_{r_g}^{r_0} dN d(r) \simeq \Delta R. \quad (20)$$

Remarkably, this result is independent of the exponent β in the size distribution; it implies that, in MH’s words, the ring “settles automatically into a state in which the gaps and the ringlets occupy comparable areas”—the geometrical optical depth is either zero (in the gaps) or much larger than unity (in the

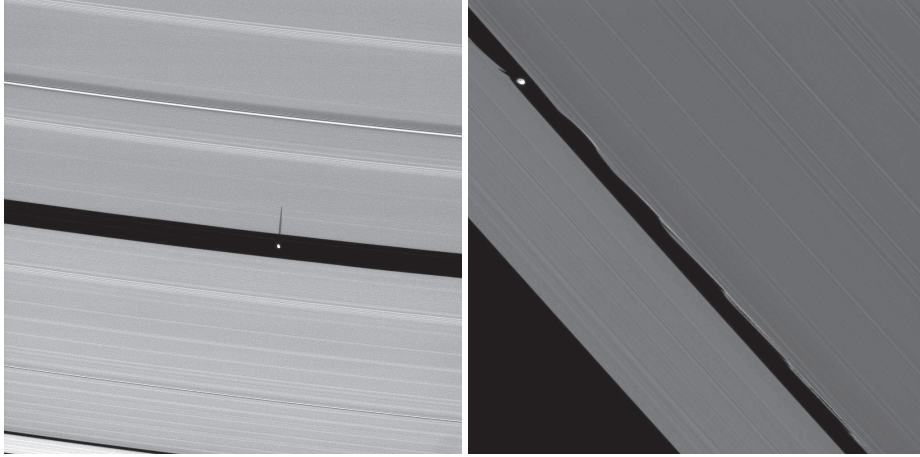


Figure 1: (left) The satellite Pan, which orbits inside the 325-km wide Encke gap in Saturn’s rings. The dark streak is Pan’s shadow, which is long because the Sun is close to the ring plane. (right) The satellite Daphnis, which orbits inside the 42-km wide Keeler gap. The periodic structures at the edges of the gap are bending waves about 1 km high, excited because Daphnis has a small orbital inclination (0.004°) relative to the rings.

ringlets between the gaps), but when averaged over scales larger than the typical gap size, as in most observations, the optical depth is of order unity.

Thus one simple assumption, that the distribution of particle sizes in the rings is a power law, led to a rich and detailed model that explained many of the then-known features of Saturn’s rings. Sadly, subsequent observations, particularly by the Cassini spacecraft since it arrived in Saturn orbit in 2004, were an example of what Thomas Huxley called “The great tragedy of science—the slaying of a beautiful hypothesis by an ugly fact”. The first problem to emerge is that the rings are not flat: in particular the satellite Mimas, which is on an orbit with an inclination of 1.57° (relative to Saturn’s equator, which is also the ring plane), excites bending waves at the 5:3 orbital resonance in the rings. The amplitude of these waves is ~ 0.5 km, large enough to explain the apparent thickness H of the edge-on ring [40]. The actual ring thickness is probably no more than a few tens of meters. A second problem is that thorough searches by Cassini have discovered only two satellites orbiting in gaps within the rings: Pan and Daphnis, with radii of 14 km and 4 km, compared to MH’s estimate of almost 10^4 satellites larger than $r_g \simeq 1.5$ km (see Figure). There is also a handful of known satellites smaller than r_g that have not cleared gaps: these include S/2009 S1, with a radius of about 0.15 km, and “propellor moonlets”, which produce S-shaped wakes a few km long. Third, observations of stellar occultations by the rings (both from the ground and spacecraft) and spacecraft

radio transmissions through the rings imply a particle size distribution with index $\beta \simeq 2$ and upper cutoff $r_{\max} \sim 10\text{--}20\text{ m}$, varying significantly with radial distance from Saturn [7]. Above this radius, the distribution of sizes steepens sharply, consistent with the small number of propellor moonlets and gap-opening satellites.

The processes that determine the distribution of sizes of Saturn’s ring particles are not understood, but apparently they do not lead to the simple power-law distribution over more than five orders of magnitude that led MH to his elegant model.

Final remarks

In preparing this retrospective, I have been forced to leave out several of my favorites among MH’s contributions: his analysis of the isochrone potential, whose simple analytic properties could have been discovered by Newton; his numerical simulations of globular cluster evolution; “Hénon’s paradox” on the escape of stars from clusters; the Hénon–Heiles potential and his work on the third integral in galactic dynamics; his comprehensive numerical study of the restricted three-body problem, which laid to rest many questions that had been unanswered for centuries; and his unexpected analytic solution of the dynamics of the Toda lattice. I hope I have captured enough of the beauty of Michel Hénon’s research that some readers will take the time to appreciate these other contributions as well.

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